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The generalized coherent states

for oscillators, connected with

Meixner and Meixner-Pollaczek polynomials ¹

Authors dedicate this work to our friend and the colleague

P.P.Kulish in connection with his 60th birthday

The investigation of the generalized coherent states for oscillator-like systems connected with given family of orthogonal polynomials is continued. In this work we consider oscillators connected with Meixner and Meixner-Pollaczek polynomials and define generalized coherent states for these oscillators. The completeness condition for these states is proved by the solution of the related classical moment problem. The results are compared with the other authors ones. In particular, we show that the Hamiltonian of the relativistic model of linear harmonic oscillator can be thought of as the linearization of the quadratic Hamiltonian which naturally arised in our formalism.

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1 Introduction

The interest to construction and investigation of coherent states for oscillator-like systems connected with a family of orthogonal polynomials has been grown in the late years (see, for example, [1]-[22]; there is the numerous bibliography in reviews [23]-[24]).

In works [25]-[35] we suggested a new method for constructions of oscillator-like systems (or, to put it briefly "oscillator"), which are connected with a family of orthogonal polynomials just as the usual boson oscillator with Hermite polynomials. This approach contains the construction of the generalized coherent states for such oscillators. In the previous works we considered the generalized coherent states for oscillators related to the classical orthogonal polynomials (Laguerre [29], Legendre [27], Chebyshev [28], Gegenbauer [30]) and the Hermite q -polynomials [32]-[34].

We describe in brief the essential points of our approach. Firstly, we put the recurrent relations for these polynomials in a symmetrical form [35] (with symmetrical Jacobi matrix) by a renormalization of these polynomials. Secondly, by standard way [35] we introduce the (generalized) coordinate and momentum operators, the associated ladder operators of creation and annihilation and *quadratic* Hamiltonian which spectrum is defined by the coefficients of (symmetrized) recurrent relations for considered polynomials. In this way we define the oscillator connected with given polynomials. Realizing the creation and annihilation operators for constructed oscillator as a differential (or a difference) operators (see [26]), we get a differential (or a difference) equation for eigenvectors of the above quadratic Hamiltonian. As a rule, (for known systems of orthogonal polynomials) this equation is equivalent to the standard second

order differential (or difference) equation for these polynomials. It is natural to consider this equation as an analogue of the Schrödinger equation for oscillator being investigated.

Next we introduce generalized coherent states of Barut-Girardello type [36]) as the eigenstates of the annihilation operator for constructed oscillator. To prove the (over)completeness of this family of states we have solve an appropriate classical moment problem. The measure which worked out this moment problem is employing in resolution of unity for the constructed family of the coherent states. For the classical polynomials (both depending on a continuous, and on a discrete variable) the arised classical moment problem is determined and its solution not need much effort. However for the deformed polynomial systems more difficult undetermined moment problem sturt up (see [55]-[56]). In this case we used the results of research of this problem obtained together with P.P.Kulish in [25].

Note that the derived explicit form of coherent states for generalized oscillators associated with orthogonal polynomials allows to calculate the values of some physically interesting quantities (such as, for example, Mandel parameter) for such systems.

If there is the dynamical symmetry group (or algebra) for constructed oscillator-like system, it is possible to define the Perelomov type coherent states in the standard way [37] (as action of the unitary shift operator on the fixed state vector). Recall that for the standard boson oscillator connected with the Heisenberg group the Barut-Girardello coherent states coincide with coherent states of the Perelomov type, as well as with the states minimizing the Heisenberg uncertainty relation. However this is not so in the more complicated cases.

It is essential for possible applications that the suggested construction of oscillator-like systems and related coherent states allows to pick up for given energy spectrum a suitable family of orthogonal polynomials (such that the coefficients of recurrent relations for these polynomials determine the spectrum of the Hamiltonian) diagonalising Hamiltonian of this system. In this way we bypass a difficult factorisation problem for Hamiltonian of this system.

Further note that together with classical orthogonal polynomials and their deformed analogues in physical researches of last years the growing attention is given to discrete polynomials (such as, for example, Hahn polynomials [38]-[39]), satisfying not a differential but a difference equation. After the publication of the works [40]-[41], where the continual analogues of such polynomials (that is the polynomials which argument is extended to continuous values, the orthogonality relation is written by an integral, and, finally, the index is continued in complex plane) was introduced, the connection of these polynomials (and also the Meixner and Meixner - Pollaczek polynomials [44]-[45]) with the Heisenberg group was founded [42]-[43]. So it is nat-

ural to investigate the oscillator-like systems defined by these polynomials. Attempts of such construction ' was undertaken in the work [10] (Meixner and Meixner-Pollaczek polynomials) and in [46] (Hahn polynomials). The Pollaczek polynomials were involved in the description [47] of the wave functions of relativistic model of linear harmonic oscillator in the frame-work of the quasi-potential approach. (For more details on this model and its variants we refer the reader to [48]-[51]).

This model was used also in [10], where Hamiltonian (which spectrum depends linearly on n) was defined. In this work for the case of Meixner and Meixner - Pollaczek polynomials Barut - Girardello coherent states are constructed and it was shown that $sp(2, \mathbb{R})$ is dynamical symmetry algebra of considered model and that the Hamiltonian is one of the generators of this algebra. That allows to define Perelomov type coherent states for this model.

Because the Meixner and Meixner-Pollaczek polynomials fulfill three-terms recurrent relation, the construction of oscillator-like systems described above is also applied to them. In the present work we shall construct these oscillator-like systems and define the generalized coherent states (both Barut - Girardello type and Perelomov type) for these systems.

We shall show, in particular, that Hamiltonian of the model, considered in [10], can be thought as a linearization of quadratic Hamiltonian, naturally arising in our approach. Note that the connection of the Meixner oscillator (at specific value of parameter $\varphi = \frac{\pi}{2}$) with the relativistic model of linear oscillator [52] was mentioned in the work [10]. However this connection was not obvious and the reasons of appearance of specific value of parameter $\varphi = \frac{\pi}{2}$ were not clear because of absence of the appropriate calculations. In the given work we shall analyze this connection and show, that only for values of parameter $\varphi = \frac{\pi}{2} + k\pi$ Hamiltonian of the Meixner oscillator coincide with Hamiltonian of the quasi-potential model of relativistic oscillator from [40].

2 Meixner oscillator

2.1 Meixner polynomials

Let us remember, that a generalized hypergeometric series ${}_rF_s$ defined by

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \frac{z^k}{k!}, \quad (1)$$

where the shifted factorials (Pochhammer symbols) are given by

$$(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \quad (2)$$

For $\beta > 0$ and $0 < \gamma < 1$ the Meixner polynomials [44]

$$M_n(\xi; \beta, \gamma) = {}_2F_1 \left(\begin{matrix} -n, -\xi \\ \beta \end{matrix} \middle| 1 - \frac{1}{\gamma} \right) = M_\xi(n; \beta, \gamma). \quad (3)$$

form two-parameter polynomial family $n = 0, 1, 2, \dots$, fulfilling the orthogonality relation

$$\sum_{m=0}^{\infty} \rho^M(m) M_n(m; \beta, \gamma) M_k(m; \beta, \gamma) = d_n \delta_{nk}, \quad (4)$$

with respect to the weight function

$$\rho^M(m)(\xi) = \frac{(\beta)_\xi \gamma^\xi}{\xi!}, \quad (5)$$

where the value of a square of norm is given by

$$d_n = \frac{n!}{\gamma^n (\beta)_n (1-\gamma)^\beta}. \quad (6)$$

These polynomials also fulfill the recurrent relations [44, 39]

$$[n + (n + \beta)\gamma - (1 - \gamma)\xi] M_n(\xi; \beta, \gamma) = (n + \beta)\gamma M_{n+1}(\xi; \beta, \gamma) + n M_{n-1}(\xi; \beta, \gamma). \quad (7)$$

The difference equation for Meixner polynomials looks like [44, 39]

$$[\gamma(\xi + \beta)e^{\partial_\xi} + \xi e^{-\partial_\xi} - (1 + \gamma)(\xi + \frac{1}{2}\beta) + (1 - \gamma)(n + \frac{1}{2}\beta)] M_n(\xi; \beta, \gamma) = 0. \quad (8)$$

The reproducing functions for Meixner polynomials have the forms

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(\xi; \beta, \gamma) t^n = \left(1 - \frac{t}{\gamma}\right)^\xi (1 - t)^{-\xi - \beta}, \quad (9)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} M_n(\xi; \beta, \gamma) t^n = e^t {}_1F_1 \left(\begin{matrix} -\xi \\ \beta \end{matrix} \middle| \left(\frac{1-\gamma}{\gamma}\right) t \right). \quad (10)$$

To get a symmetrical form of recurrent relations (7) we (following [35]) define the renormalized Meixner polynomials

$$\widetilde{M}_n(\xi; \beta, \gamma) = \frac{M_n(\xi; \beta, \gamma)}{c_n}, \quad (11)$$

with

$$c_0 = 1, \quad c_n = \gamma^{-n/2} \sqrt{\frac{n!}{(\beta)_n}}, \quad n \geq 1, \quad (12)$$

and where we take into account, that $M_0(\xi; \beta, \gamma) = 1$. Then polynomials $\widetilde{M}_n(\xi; \beta, \gamma)$ satisfy canonical three-term recurrent relations with symmetrical Jacobi matrix

$$\xi \widetilde{M}_n(\xi; \beta, \gamma) = b_n \widetilde{M}_{n+1}(\xi; \beta, \gamma) - a_n \widetilde{M}_n(\xi; \beta, \gamma) + b_{n-1} \widetilde{M}_{n-1}(\xi; \beta, \gamma) \quad (13)$$

$$\widetilde{M}_0(\xi; \beta, \gamma) = 1, \quad (14)$$

which coefficients are defined by the formulas

$$b_{-1} = 0, \quad b_n = \frac{\sqrt{\gamma}}{\gamma - 1} \sqrt{(\beta + n)(n + 1)}, \quad a_n = \frac{n + (n + \beta)\gamma}{\gamma - 1}, \quad n \geq 0. \quad (15)$$

Let us remark that Jacobi matrix defined by the relation (13) has a nonzero diagonal and the appropriate moment problem is determined [25].

For normalized Meixner polynomials $\widetilde{M}_n(\xi; \beta, \gamma)$ the orthogonality relation (4) takes the form

$$\sum_{\xi=0}^{\infty} \widetilde{\rho}^M(\xi) \widetilde{M}_n(\xi; \beta, \gamma) \widetilde{M}_m(\xi; \beta, \gamma) = \delta_{nm} \quad (16)$$

with the weight function

$$\widetilde{\rho}^M(\xi) = \frac{(\beta)_\xi \gamma^\xi}{\xi!} (1 - \gamma)^\beta. \quad (17)$$

The left hand side of the relation (16) can be rewritten as integral over the discrete measure with carriers in the points $\xi = 0, 1, 2, \dots$ and the loadings given by (17).

Now we define Meixner functions

$$\psi_n^M(\xi; \beta, \gamma) = (-1)^n \sqrt{\widetilde{\rho}^M(\xi)} \widetilde{M}_n(\xi; \beta, \gamma), \quad (18)$$

which satisfy in the space $\mathcal{H}_0^M := \ell^2$ the discrete orthogonality relations

$$\sum_{\xi=0}^{\infty} \psi_n^M(\xi; \beta, \gamma) \psi_m^M(\xi; \beta, \gamma) = \delta_{nm}, \quad (19)$$

or (in view of symmetry of Meixner polynomials (3)) the orthogonality relations

$$\sum_{n=0}^{\infty} \psi_n^M(\xi; \beta, \gamma) \psi_m^M(\xi'; \beta, \gamma) = \delta_{\xi\xi'}. \quad (20)$$

2.2 Meixner oscillator

We will introduce the Hilbert space $\mathcal{H}^M := \ell^2(\widetilde{\rho}^M)$ with weight function $\widetilde{\rho}^M$ (17) and the basis $\{\widetilde{M}_n(\xi; \beta, \gamma)\}_{n=0}^{\infty}$. Let $\widehat{\mathcal{H}}^M$ be the same space \mathcal{H}^M with the basis $\{\widehat{M}_n(\xi; \beta, \gamma)\}_{n=0}^{\infty}$, where

$$\widehat{M}_n(\xi; \beta, \gamma) = (-1)^n \widetilde{M}_n(\xi; \beta, \gamma).$$

The polynomials \widehat{M}_n satisfy the recurrent relations

$$(-\xi)\widehat{M}_n(\xi; \beta, \gamma) = b_n\widehat{M}_{n+1}(\xi; \beta, \gamma) + a_n\widehat{M}_n(\xi; \beta, \gamma) + b_{n-1}\widehat{M}_{n-1}(\xi; \beta, \gamma). \quad (21)$$

Further we define urther the following unitary transformations

$$U_1 : \mathcal{H}_0^M \rightarrow \widehat{\mathcal{H}}^M \quad \Rightarrow \quad U_1\varphi = \frac{\varphi}{\sqrt{\widehat{\rho}^M}} = \widehat{\varphi} \in \widehat{\mathcal{H}}^M; \quad (22)$$

$$U_2 : \widehat{\mathcal{H}}^M \rightarrow \mathcal{H}^M \quad \Rightarrow \quad U_2\widehat{M}_n = \widetilde{M}_n, \quad n = 0, 1, 2, \dots; \quad (23)$$

$$U : \mathcal{H}_0^M \rightarrow \mathcal{H}^M \quad \Rightarrow \quad U = U_2 \circ U_1. \quad (24)$$

Following [35], we define "coordinate" X and momentum P operators by their action on elements of basis $\{\widetilde{M}_n(\xi; \beta, \gamma)\}_{n=0}^{\infty}$ in Hilbert space \mathcal{H}^M according to formulas

$$\begin{aligned} X\widetilde{M}_0(\xi; \beta, \gamma) &= b_0\widetilde{M}_1(\xi; \beta, \gamma) - a_0\widetilde{M}_0(\xi; \beta, \gamma), \\ X\widetilde{M}_n(\xi; \beta, \gamma) &= b_n\widetilde{M}_{n+1}(\xi; \beta, \gamma) - a_n\widetilde{M}_n(\xi; \beta, \gamma) + b_{n-1}\widetilde{M}_{n-1}(\xi; \beta, \gamma), \quad n \geq 1; \end{aligned} \quad (25)$$

$$\begin{aligned} P\widetilde{M}_0(\xi; \beta, \gamma) &= -ib_0\widetilde{M}_1(\xi; \beta, \gamma) - a_0\widetilde{M}_0(\xi; \beta, \gamma), \\ P\widetilde{M}_n(\xi; \beta, \gamma) &= i(b_{n-1}\widetilde{M}_{n-1}(\xi; \beta, \gamma) - b_n\widetilde{M}_{n+1}(\xi; \beta, \gamma)) - a_n\widetilde{M}_n(\xi; \beta, \gamma), \quad n \geq 1. \end{aligned} \quad (26)$$

Now we define oscillator-like system, which we shall name *Meixner oscillator*, by introducing the generalized coordinate \widetilde{X} and the generalized momentum \widetilde{P}

$$\widetilde{X} := \text{Re}(X - P), \quad \widetilde{P} := -i\text{Im}(X - P), \quad (27)$$

and creation and annihilation operators

$$\widetilde{a}^+ := \frac{1}{\sqrt{2}}(\widetilde{X} + i\widetilde{P}), \quad \widetilde{a}^- := \frac{1}{\sqrt{2}}(\widetilde{X} - i\widetilde{P}). \quad (28)$$

The Hamiltonian of the Meixner oscillator we choose in the form

$$\widetilde{H}^M = \widetilde{X}^2 + \widetilde{P}^2. \quad (29)$$

It is follows from the work [35] that a spectrum of the Hamiltonian looks like

$$\begin{aligned} \lambda_0 &= 2b_0^2 = 2\beta \left(\frac{\sqrt{\gamma}}{\gamma-1} \right)^2; \\ \lambda_n &= 2(b_{n-1}^2 + b_n^2) = \left(\frac{2\sqrt{\gamma}}{\gamma-1} \right)^2 (n^2 + n\beta + \frac{1}{2}\beta), \quad n \geq 1. \end{aligned} \quad (30)$$

Note that it is possible to define in the Hilbert space $\widehat{\mathcal{H}}^M$ the Meixner oscillator connected with polynomials \widehat{M}_n by similar considerations. However, from (25)-(29) it follows that these oscillators coincide.

In just the same way as in [35] it is possible to show that the eigenvalue equation $\widetilde{H}^M y = \lambda_n y$ is equivalent to the difference equation

$$n(\gamma - 1)y(\xi) = \gamma(\xi + \beta)y(\xi + 1) - [\xi + (\xi + \beta)\gamma]y(\xi) + \xi y(\xi - 1), \quad y(\xi) = \widetilde{M}_n(\xi; \beta, \gamma), \quad (31)$$

which it is natural to call the Schrödinger equation of constructed Meixner oscillator. (Note that the equation (31) is another form of the equation (8)).

We point out one essential difference of our Meixner oscillator from ones considered in [10]. The Meixner oscillator Hamiltonian from [10] in the space \mathcal{H}_0^M has the following form

$$\overset{\circ}{H}^M(\xi) = \frac{1+\gamma}{1-\gamma} \left(\xi + \frac{1}{2}\beta \right) - \frac{\sqrt{\gamma}}{1-\gamma} [\mu(\xi)e^{\partial_\xi} + \mu(\xi-1)e^{-\partial_\xi}], \quad \mu(\xi) = \sqrt{(\xi+1)(\xi+\beta)}. \quad (32)$$

Its eigenvalues are linear in n

$$\mu_n = n + \frac{1}{2}\beta, \quad n = 0, 1, 2, \dots, \quad (33)$$

whereas the spectrum of the Hamiltonian \widetilde{H}^M has quadratic dependence on n (see (30)) in our case. The eigenvalue equation of Hamiltonian $\overset{\circ}{H}^M(\xi)$ (32) can be written as a difference equation (coinciding with (8) and (31))

$$\left[\gamma(\xi + \beta)e^{\partial_\xi} + \xi e^{-\partial_\xi} - (1 + \gamma)(\xi + \frac{1}{2}\beta) + (1 - \gamma)(n + \frac{1}{2}\beta) \right] M_n(\xi; \beta, \gamma) = 0. \quad (34)$$

3 Dynamic symmetry algebra and connection of Hamiltonians \widetilde{H} and H

In the Hilbert space \mathcal{H}^M we shall define operators

$$K_+^M := \frac{1-\gamma}{\sqrt{2\gamma}} \widetilde{a}^+, \quad K_-^M := \frac{1-\gamma}{\sqrt{2\gamma}} \widetilde{a}^-, \quad K_0^M := \frac{1}{2} [K_-^M, K_+^M], \quad (35)$$

and define a new Hamiltonian $H^M = U \overset{\circ}{H}^M U^{-1} = K_0^M$. It is possible to show that operators K_+^M, K_-^M, K_0^M give realization of commutation relations

$$[K_0^M, K_\pm^M] = \pm K_\pm^M, \quad [K_-^M, K_+^M] = 2K_0^M; \quad (36)$$

of $sp(2, \mathbb{R})$ algebra.

In the work [10] the following realization of generators $\overset{\circ}{K}_+^M = U^{-1}K_+^MU$ and $\overset{\circ}{K}_-^M = U^{-1}K_-^MU$ as difference operators (in space $\overset{\circ}{\mathcal{H}}^M$) was given

$$\overset{\circ}{K}_+^M = \frac{\gamma}{1-\gamma}\mu(\xi)e^{\partial_\xi} + \frac{1}{1-\gamma}e^{-\partial_\xi}\mu(\xi) - \frac{2\sqrt{\gamma}}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right), \quad (37)$$

$$\overset{\circ}{K}_-^M = \frac{\gamma}{1-\gamma}\mu(\xi)e^{\partial_\xi} + \frac{\gamma}{1-\gamma}e^{-\partial_\xi}\mu(\xi) - \frac{2\sqrt{\gamma}}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right), \quad (38)$$

Let $\widehat{K}_\pm^M = U_2^{-1}K_\pm^MU_2$ and $\widehat{H}^M = U_2^{-1}H^MU_2$ are the generators of algebra $sp(2, \mathbb{R})$ in the space $\widehat{\mathcal{H}}^M$, which are connected to generators from [10] by unitary transformation

$$\widehat{K}_+^M = U_1\overset{\circ}{K}_+^MU_1^{-1}, \quad \widehat{K}_-^M = U_1\overset{\circ}{K}_-^MU_1^{-1}, \quad \widehat{H}^M = U_1H_0^MU_1^{-1}. \quad (39)$$

From (32), (37) and (38) it follows that

$$\widehat{K}_+^M = \frac{\gamma^{3/2}}{1-\gamma}(\xi + \beta)e^{\partial_\xi} + \frac{1}{\sqrt{\gamma}(1-\gamma)}\xi e^{-\partial_\xi} - \frac{2\sqrt{\gamma}}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right), \quad (40)$$

$$\widehat{K}_-^M = \frac{\sqrt{\gamma}}{1-\gamma}(\xi + \beta)e^{\partial_\xi} + \frac{\sqrt{\gamma}}{1-\gamma}\xi e^{-\partial_\xi} - \frac{2\sqrt{\gamma}}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right), \quad (41)$$

$$\widehat{H}^M = \frac{1+\gamma}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right) - \frac{\gamma}{1-\gamma}(\xi + \beta)e^{\partial_\xi} - \frac{1}{1-\gamma}\xi e^{-\partial_\xi}. \quad (42)$$

On elements \widehat{M}_n of the basis in the space $\widehat{\mathcal{H}}^M$ these operators act by the following formulas

$$\widehat{K}_+^M\widehat{M}_n = \mu(n)\widehat{M}_{n+1}, \quad \widehat{K}_-^M\widehat{M}_n = \mu(n-1)\widehat{M}_{n-1}, \quad \widehat{H}^M\widehat{M}_n = \left(n + \frac{1}{2}\beta\right)\widehat{M}_n, \quad (43)$$

and on elements \widetilde{M}_n of the basis in the space $\widetilde{\mathcal{H}}$ by

$$K_+^M\widetilde{M}_n = -\mu(n)\widetilde{M}_{n+1}, \quad K_-^M\widetilde{M}_n = -\mu(n-1)\widetilde{M}_{n-1}, \quad H^M\widetilde{M}_n = \left(n + \frac{1}{2}\beta\right)\widetilde{M}_n. \quad (44)$$

The cartesian generators K_1^M, K_2^M of the algebra $sp(2, \mathbb{R})$ and the Cazimir operator \mathcal{C}_2 are defined by the relations

$$\widehat{K}_1^M = -\frac{i}{2}(\widehat{K}_+^M - \widehat{K}_-^M) = \frac{i}{2}(\mu(\xi)e^{\partial_\xi} - \mu(\xi)e^{-\partial_\xi}\mu(\xi)), \quad (45)$$

$$\widehat{K}_2^M = -\frac{1}{2}(\widehat{K}_+^M + \widehat{K}_-^M) = -\frac{1+\gamma}{2(1-\gamma)}(\mu(\xi)e^{\partial_\xi} + \mu(\xi)e^{-\partial_\xi}\mu(\xi)) + \frac{2\sqrt{\gamma}}{1-\gamma}\left(\xi + \frac{1}{2}\beta\right), \quad (46)$$

$$\widehat{\mathcal{C}}_2 = (\widehat{K}^M)^2 = (\widehat{K}_0^M)^2 - (\widehat{K}_1^M)^2 - (\widehat{K}_2^M)^2 = (\widehat{K}_0^M)^2 - \widehat{K}_0^M - \widehat{K}_+^M\widehat{K}_-^M = \frac{1}{2}\beta\left(\frac{1}{2}\beta - I\right). \quad (47)$$

Selfadjoint Hamiltonians $\widehat{\widetilde{H}}^M = U_2^{-1}\widetilde{H}^MU_2$ and H^M in the Hilbert space $\widehat{\mathcal{H}}^M$ have the same set of eigenfunctions $\left\{\widehat{M}_n(\xi; \beta, \gamma)\right\}_{n=0}^\infty$ and are connected with each other by the relation

$$\widehat{\widetilde{H}}^M = \frac{4\gamma}{(\gamma-1)^2}\left((\widehat{H}^M)^2 - (\widehat{K}^M)^2\right). \quad (48)$$

Hence, the self-adjoint Hamiltonians \tilde{H}^M and H^M in the Hilbert space \mathcal{H}^M are connected with each other by same relation

$$\tilde{H}^M = \frac{4\gamma}{(\gamma-1)^2} ((H^M)^2 - (K^M)^2). \quad (49)$$

This formula gives an interesting connection of our Hamiltonian with Hamiltonian for generalized harmonic oscillator model in the approach used in [10].

4 Barut - Girardello coherent states for Meixner oscillator

4.1 Definition of coherent states

By definition, Barut - Girardello [36] coherent states for Meixner oscillator are eigenstates of annihilation operator \tilde{a}^- . Let $|n\rangle = \tilde{M}_n(\xi; \beta, \gamma)$ denote the elements of the Fock basis (the oscillator basis) in Fock Hilbert space (the space of filling numbers) \mathcal{H}^M . Then we have

$$\tilde{a}^-|z\rangle = z|z\rangle, \quad |z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{(\sqrt{2}b_{n-1})!} |n\rangle, \quad (50)$$

where coefficients b_n are determined by the formula (15) and we introduce the notation

$$(\sqrt{2}b_{-1})! = 1, \quad (\sqrt{2}b_n)! = 2^{\frac{1}{2}n} b_0 b_1 \cdots b_{n-1}, \quad n \geq 1. \quad (51)$$

For normalizing factor $\mathcal{N}(|z|^2)$ one obtains

$$\mathcal{N}^2(|z|^2) = \langle z|z\rangle = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2b_{n-1}^2)!} = \sum_{n=0}^{\infty} \left(\frac{(1-\gamma)^2}{2\gamma} \right)^n \frac{|z|^{2n}}{(\beta)_n n!}. \quad (52)$$

Because the series in (52) has infinite convergence radius $R = \infty$, this series converges on all complex plane.

From the formula (15) it follows that

$$(2b_{n-1}^2)! = \left(\frac{2\gamma}{(1-\gamma)^2} \right)^n (\beta)_n n! = \left(\frac{2\gamma}{(1-\gamma)^2} \right)^n \frac{n! \Gamma(\beta + n)}{\Gamma(\beta)}. \quad (53)$$

Taking into account an explicit expression for Bessel function of the 1-kind

$$I_\alpha(2\sqrt{z}) := z^{\frac{1}{2}\alpha} \sum_{m=0}^{\infty} \frac{z^m}{m! \Gamma(m + \alpha + 1)}, \quad (54)$$

we receive

$$\mathcal{N}^2(|z|^2) = \left(\frac{(1-\gamma)^2}{2\gamma} \right)^{\frac{1}{2}(\beta-1)} \frac{\Gamma(\beta)}{|z|^{\beta-1}} I_{\beta-1} \left(2 \frac{(1-\gamma)}{\sqrt{2\gamma}} |z| \right). \quad (55)$$

It allows to rewrite expression (50) for the coherent state in the following way

$$|z\rangle = \frac{\left(\frac{(1-\gamma)^2}{2\gamma} |z| \right)^{\frac{1}{2}(\beta-1)}}{\left[I_{\beta-1} \left(2 \frac{(1-\gamma)}{\sqrt{2\gamma}} |z| \right) \Gamma(\beta) \right]^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{M_n(\xi; \beta, \gamma)}{n!} \left(\frac{1-\gamma}{\sqrt{2\gamma}} z \right)^n. \quad (56)$$

Using the reproducing function (10) for Meixner polynomials, we can evaluate the series in the formula (56). This allows to obtain explicit expression for Barut - Girardello coherent state of Meixner oscillator

$$|z\rangle = \sqrt{\frac{1-\gamma}{\sqrt{2\gamma}}} |z|^{\frac{1}{2}(\beta-1)} \left[\Gamma(\beta) I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z| \right) \right]^{-\frac{1}{2}} \exp \left[\frac{1-\gamma}{\sqrt{2\gamma}} z \right] {}_1F_1 \left(\frac{-\xi}{\beta} \left| \frac{(1-\gamma)^2}{\gamma\sqrt{2\gamma}} \right. z \right). \quad (57)$$

For overlapping of two coherent states we obtain

$$\langle z_1 | z_2 \rangle = I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} \sqrt{z_1 z_2} \right) \left[I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z_1| \right) I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z_2| \right) \right]^{-\frac{1}{2}}. \quad (58)$$

Note, that for $\alpha = \beta - 1$ and $\gamma = 2 - \sqrt{3}$ this expression coincides with similar one for coherent states of Laguerre oscillator (see the formula (24) from [31]). Note also that taking into account the relation (17) for Meixner function, we obtain

$$\psi_n^M(\xi; \beta, \gamma) = (-1)^n \psi_0(\xi; \beta, \gamma) \widetilde{M}_n(\xi; \beta, \gamma), \quad \psi_0^M(\xi; \beta, \gamma) = \sqrt{\frac{(\beta)_\xi \gamma^\xi}{\xi!} (1-\gamma)^\beta}. \quad (59)$$

So our expression for the coherent state is in agreement with the relation (69) from Atakishiev a.o. work [10], (if we take into account, that in work [10] are considered not normalized coherent states).

4.2 Proof of the (over)completeness for constructed family of coherent states

The most important property of the family of coherent states is the (over)completeness property that can be expressed as validity of the resolution of unity relation

$$\iint_{\mathbb{C}} |z\rangle \langle z| \widehat{W}(|z|^2) d^2 z = \mathbb{1}. \quad (60)$$

To check this formula it is necessary to construct a measure

$$d\mu(|z|^2) = \widehat{W}(|z|^2) d^2 z. \quad (61)$$

It is known [12] - [15] that for this purpose we have to solve the Stieltjes classical moment problem [55, 56]

$$\pi \int_0^\infty x^n W(x) dx = \left(\frac{2\gamma}{(1-\gamma)^2} \right)^n \frac{n! \Gamma(\beta+n)}{\Gamma(\beta)}, \quad n \geq 0, \quad (62)$$

where

$$W(x) = \frac{\widehat{W}(x)}{\mathcal{N}^2(x)}, \quad (x = |z|^2). \quad (63)$$

Taking into account the formula (6.561.16) from [54], we find

$$W(x) = \frac{2}{\pi} \frac{(1-\gamma)^2}{2\gamma\Gamma(\beta)} \left(\frac{(1-\gamma)^2}{2\gamma} x \right)^{\frac{1}{2}(\beta-1)} K_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} \sqrt{x} \right). \quad (64)$$

Then

$$\widehat{W}(|z|^2) = \frac{(1-\gamma)^2}{\pi\gamma} K_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z| \right) I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z| \right), \quad (65)$$

and we obtain for a measure (61) the expression

$$d\mu(|z|^2) = \frac{(1-\gamma)^2}{\pi\gamma} K_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z| \right) I_{\beta-1} \left(2 \frac{1-\gamma}{\sqrt{2\gamma}} |z| \right) d^2 z. \quad (66)$$

For $\alpha = \beta - 1$ and $\gamma = 2 - \sqrt{3}$ this result coincides with similar expression for the case of Laguerre polynomials [31].

5 The Perelomov coherent states for Meixner oscillator

The Perelomov coherent states connected with dynamical algebra $su(1|1)$ in a case of the Meixner oscillator can be defined by

$$|\zeta\rangle = (1 - |\zeta|^2)^{\frac{1}{2}\beta} \exp(\zeta K_+^M) \widetilde{M}_0(\xi, \beta, \gamma) = (1 - |\zeta|^2)^{\frac{1}{2}\beta} \sum_{n=0}^{\infty} \sqrt{\frac{(\beta)_n}{n!}} \widetilde{M}_n(\xi, \beta, \gamma) \zeta^n, \quad (67)$$

where $\zeta \in \mathbb{C}$ and $|\zeta| < 1$. Using the reproducing function (10) for Meixner polynomials and taking into account relations (11)-(12), we find

$$\begin{aligned} |\zeta\rangle &= (1 - |\zeta|^2)^{\frac{1}{2}\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(\xi, \beta, \gamma) (\sqrt{\gamma}\zeta)^n \\ &= (1 - |\zeta|^2)^{\frac{1}{2}\beta} \left(1 - \frac{\sqrt{\gamma}\zeta}{\gamma} \right)^\xi (1 - \sqrt{\gamma}\zeta)^{\xi-\beta}. \end{aligned} \quad (68)$$

This result coincides with the relation (75) in [10]. For overlapping of coherent states (68) we receive the expression

$$\langle \zeta_1 | \zeta_2 \rangle = [(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)]^{\frac{1}{2}\beta} (1 - \overline{\zeta_1}\zeta_2)^{-\beta}, \quad (69)$$

which also coincides with the similar result from [10].

To prove the overcompleteness property it is necessary to find a measure from resolution of unity for these states. It brings us to a moment problem

$$\pi \int_0^\infty x^n W(x) dx = \frac{n!}{(\beta)_n}, \quad n = 0, 1, \dots \quad (70)$$

Using the formula (3.251.1) from [54], we receive

$$W(x) = \frac{\beta - 1}{\pi} (1 - x)^{\beta - 2}. \quad (71)$$

Hence the measure (61) looks like

$$d\mu(|\zeta|^2) = \frac{\beta - 1}{\pi} \frac{d^2\zeta}{(1 - |\zeta|^2)^2}. \quad (72)$$

For $\beta = \alpha + 1$ this result also conform with the similar relation for the case of Laguerre polynomials.

We stress that argument ζ of Perelomov coherent states belongs to an interior of unit circle on complex plane $|\zeta| < 1$, where as in the case of Barut - Girardello coherent states one has $z \in \mathbb{C}$.

6 Meixner - Pollaczek oscillator and its coherent states

6.1 Meixner - Pollaczek polynomials

Meixner - Pollaczek polynomials [39] are defined by hypergeometric function:

$$P_n^\nu(\xi; \varphi) = \frac{(2\nu)_n}{n!} e^{in\varphi} {}_2F_1 \left(\begin{matrix} -n, \nu + i\xi \\ 2\nu \end{matrix} \middle| 1 - e^{-2i\varphi} \right), \quad (73)$$

and connected with Meixner polynomials (3) by the relation

$$P_n^\nu(\xi; \varphi) = \frac{e^{-in\varphi}}{n!} (2\nu)_n M_n(i\xi - \nu; 2\nu, e^{-2i\varphi}). \quad (74)$$

The orthogonality relation for Meixner - Pollaczek polynomials has the form ($\nu > 0$, $0 < \varphi < \pi$)

$$\frac{1}{2\pi} \int_{-\infty}^\infty e^{(2\varphi - \pi)\xi} |\Gamma(\nu + i\xi)|^2 P_m^\nu(\xi; \varphi) P_n^\nu(\xi; \varphi) d\varphi = \frac{\Gamma(n + 2\nu)}{(2 \sin \varphi)^{2\nu} n!} \delta_{m,n}. \quad (75)$$

These polynomials satisfy the recurrent relations

$$\xi P_n^\nu(\xi; \varphi) = \tilde{b}_n P_{n+1}^\nu(\xi; \varphi) - \tilde{a}_n P_n^\nu(\xi; \varphi) + \tilde{c}_n P_{n-1}^\nu(\xi; \varphi); \quad P_0^\nu(\xi; \varphi) = 1, \quad (76)$$

where

$$\tilde{a}_n = (n + \nu) \operatorname{ctg} \varphi, \quad \tilde{b}_n = \frac{n + 1}{2 \sin \varphi}, \quad \tilde{c}_n = \frac{n + 2\nu - 1}{2 \sin \varphi}, \quad (77)$$

and have the following symmetry property

$$P_n^\nu(-\xi; -\varphi) = P_n^\nu(\xi; \varphi), \quad (78)$$

Renormalized (according [35]) polynomials

$$\widehat{P}_n^\nu(\xi; \varphi) = P_n^\nu(\xi; \varphi) \sqrt{\frac{n!}{(2\nu)_n}}, \quad n \geq 0; \quad (79)$$

satisfy the symmetric recurrent relations

$$\xi \widehat{P}_n^\nu(\xi; \varphi) = \alpha_n \widehat{P}_{n+1}^\nu(\xi; \varphi) - \tilde{a}_n \widehat{P}_n^\nu(\xi; \varphi) + \alpha_{n-1} \widehat{P}_{n-1}^\nu(\xi; \varphi); \quad \widehat{P}_0^\nu(\xi; \varphi) = 1, \quad (80)$$

where

$$\alpha_n = \frac{\sqrt{(n+1)(2\nu+n)}}{2 \sin \varphi}. \quad (81)$$

For renormalized Meixner - Pollaczek polynomials the orthogonality relation becomes

$$\int_{-\infty}^{\infty} \widehat{P}_m^\nu(\xi; \varphi) \widehat{P}_n^\nu(\xi; \varphi) \mu(d\xi) = \delta_{m,n}, \quad (82)$$

where

$$\mu(d\xi) = \frac{|\Gamma(\nu + i\xi)|^2}{2\pi\Gamma(2\nu)} e^{(2\varphi - \pi)\xi} (2 \sin \varphi)^{2\nu} d\xi \quad (83)$$

Let \mathcal{H}^P be the Hilbert space

$$\mathcal{H}^P = L^2(\mathbb{R}, \mu(d\xi)). \quad (84)$$

In what follows we shall consider unitary transformation $\widehat{V} : \widehat{\mathcal{H}}^M \rightarrow \mathcal{H}^P$. Note that the parameter γ included in recurrent relations (13) and (21) (for \widetilde{M}_n and \widehat{M}_n , accordingly) is equal to $e^{-2i\varphi}$. So for $\sqrt{\gamma} = \sqrt{e^{-2i\varphi}}$ it is possible to choose two values

$$\pm \sqrt{\gamma} = \mp e^{-i\varphi}. \quad (85)$$

From (21) choosing a minus sign, we obtain a recurrent relations

$$\xi \widehat{M}_n^-(\xi; \beta, \gamma) = -i\alpha_n \widehat{M}_{n+1}^-(\xi; \beta, \gamma) - (i\tilde{a}_n + \nu) \widehat{M}_n^-(\xi; \beta, \gamma) - i\alpha_{n-1} \widehat{M}_{n-1}^-(\xi; \beta, \gamma); \quad (86)$$

and from (13) choosing a plus sign, we obtain a recurrent relations

$$\xi \widehat{M}_n^+(\xi; \beta, \gamma) = i\alpha_n \widehat{M}_{n+1}^+(\xi; \beta, \gamma) - (i\tilde{a}_n + \nu) \widehat{M}_n^+(\xi; \beta, \gamma) + i\alpha_{n-1} \widehat{M}_{n-1}^+(\xi; \beta, \gamma). \quad (87)$$

These relations differs by a choice of value $\sqrt{e^{-2i\varphi}}$, $(0 < \varphi < \pi)$.

Taking into account (11), from (74) it follows that

$$\widehat{P}_n^\nu(\xi, \varphi) = \widetilde{M}_n^-(i\xi - \nu, 2\nu, e^{-2i\varphi}).$$

Than we can define unitary transformation \widehat{V} by the relation

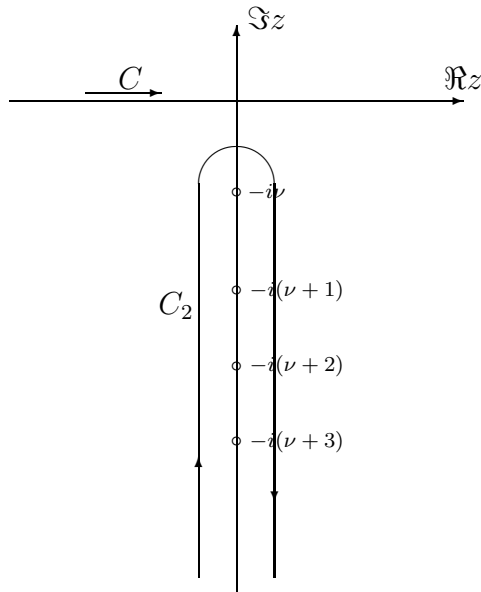
$$\widehat{P}_n^\nu(\xi, \varphi) = \widetilde{M}_n^-(i\xi - \nu, 2\nu, e^{-2i\varphi}) = (-1)^n \widehat{M}_n^-(i\xi - \nu, 2\nu, e^{-2i\varphi}) = \widehat{V} \widehat{M}_n^-(\xi, 2\nu, e^{-2i\varphi}) \quad (88)$$

Note that we choose in (85) a minus sign, appropriate to polynomials \widehat{M}_n^- , which satisfy recurrent relations (86), because the choice of a plus sign (i.e. polynomials \widehat{M}_n^+ , satisfying the recurrent relations (87)) implies a contradiction. Indeed, in this case from (87) follows $\widehat{M}_n^+(\xi, 2\nu, e^{-2i\varphi}) = \widetilde{M}_n^-(\xi, 2\nu, e^{-2i\varphi})$ and then

$$\widehat{V} \widehat{M}_n^+(\xi, 2\nu, e^{-2i\varphi}) = (-1)^n \widetilde{M}_n^-((i\xi - \nu), 2\nu, e^{-2i\varphi}) \neq \widetilde{M}_n^-((i\xi - \nu), 2\nu, e^{-2i\varphi}).$$

It is relevant to note that transition from discrete orthogonality relations (16) for Meixner polynomials to continuous orthogonality relations (82) for Meixner - Pollaczek polynomials is analogues to Zommerfeld - Watson transformation in the scattering theory. To proving the unitarity of transformation \widehat{V} we substitute (88) in orthogonality relation (82)

$$I = \int_{-\infty}^{\infty} \widehat{M}_n(\pm i\xi - \nu; 2\nu; e^{-2i\varphi}) \widehat{M}_m(\pm i\xi - \nu; 2\nu; e^{-2i\varphi}) \mu(d\xi) = \delta_{m,n}. \quad (89)$$



Note that $|\Gamma(\nu + i\xi)|^2 = \Gamma(\nu + i\xi)\Gamma(\nu - i\xi)$ and integrand in the left part of the formula (89) is analytical function on all complex plane except the points $\zeta_k = -i(k + \nu)$, $(k \geq 0)$ on the imaginary axis in which it has poles of the first order. Taking into account the asymptotic behaviour of Gamma-function at infinity we can replace a contour of integration C_2 by a contour $\mathcal{L} = \bigcup_{k=0}^{\infty} l_k$, where l_k - a circle with the centre in a point ζ_k and a radius $r \leq \frac{1}{2}$. Then, applying the residue theorem, we received

$$I = \sum_{k=0}^{\infty} \oint_{l_k} F(\zeta; \nu; \varphi) d\varphi = 2\pi i \sum_{k=0}^{\infty} \text{Res} F \Big|_{\zeta=\zeta_k}. \quad (90)$$

Because

$$\text{Res} (\Gamma(\nu - i\zeta)) \Big|_{\zeta=\zeta_k} = \frac{(-1)^k}{k!} i, \quad k = 0, 1, \dots, \quad (91)$$

and other factors in F are analytical functions, we obtain (16) from (90). Thus unitarity of transformation $V^{-1} : \mathcal{H}_P \rightarrow \mathcal{H}_M$ is proved (compare with [40]).

6.2 Meixner - Pollaczek oscillator

As above, we define in the Hilbert space $\mathcal{H}^P := L^2(\mathbb{R}, \mu(d\xi))$, considered as a Fock space, coordinate and momentum operators, ladder operators \tilde{a}^- and \tilde{a}^+ , and also Hamiltonian \tilde{H}^P . Spectrum of the Hamiltonian \tilde{H}^P is given by

$$\lambda_0 = \frac{\nu}{2 \sin^2 \varphi}, \quad \lambda_n = \frac{n(n + 2\nu)}{\sin^2 \varphi}. \quad (92)$$

Note that the eigenvalues equation $\tilde{H}^P y = \lambda y$ for the operator \tilde{H}^P is equivalent to the difference equation for Meixner - Pollaczek polynomials

$$e^{i\varphi}(\nu - i\xi)y(\xi + i) + 2i(\xi \cos \varphi - (n + \nu) \sin \varphi)y(\xi) - e^{-i\varphi}(\nu + i\xi)y(\xi - i) = 0. \quad (93)$$

As above, we can introduce another Hamiltonian, connected with relativistic oscillator [40]. For this purpose in the Hilbert space \mathcal{H}^P we define operators

$$K_+^P = \sqrt{2} \sin \varphi \tilde{a}_P^+, \quad K_-^P = \sqrt{2} \sin \varphi \tilde{a}_P^-, \quad K_0^P = \frac{1}{2} [K_-^P, K_+^P] = H^P, \quad (94)$$

which fulfill the commutation relations of $sp(2, \mathbb{R})$ algebra

$$[K_-^P, K_+^P] = 2K_0^P, \quad [K_0^P, K_{\pm}^P] = \pm K_{\pm}^P. \quad (95)$$

The Casimir operator takes the form

$$K^2 = (K_0^P)^2 - (K_1^P)^2 - (K_2^P)^2, \quad (96)$$

where

$$K_1^P = -\frac{i}{2}(K_+^P - K_-^P), \quad K_2^P = -\frac{i}{2}(K_+^P + K_-^P). \quad (97)$$

Using the results of work [35], we receive from relations (94), (35), (44) and (74)

$$K_+^P = \widehat{V} \widehat{K}_+^M \widehat{V}^{-1}, \quad K_-^P = \widehat{V} \widehat{K}_-^M \widehat{V}^{-1}, \quad H^P = \widehat{V} \widehat{H}^M \widehat{V}^{-1}. \quad (98)$$

Recall that from (81) and (15) for $\gamma = e^{-2i\varphi}$ $\beta = 2\nu$ we obtain

$$b_n^\pm = \mp i\alpha_n, \quad a_n^\pm = i\tilde{a}_n + \nu. \quad (99)$$

Then from recurrent relations (13) and (86) we have

$$\begin{aligned} & \left(\widehat{V} \xi \widehat{V}^{-1} \right) \widehat{P}_n^\nu(\xi; \varphi) = \widehat{V} \xi \widehat{M}_n^-(\xi, 2\nu, e^{-2i\varphi}) = \\ & = \widehat{V} \left(-i\alpha_n \widehat{M}_{n+1}^-(\xi, 2\nu, e^{-2i\varphi}) - (i\tilde{a}_n + \nu) \widehat{M}_n^-(\xi, 2\nu, e^{-2i\varphi}) - i\alpha_{n-1} \widehat{M}_{n-1}^-(\xi, 2\nu, e^{-2i\varphi}) \right) = \\ & = -i \left(\alpha_n \widehat{P}_{n+1}^\nu(\xi; \varphi) - \tilde{a}_n \widehat{P}_n^\nu(\xi; \varphi) + \alpha_{n-1} \widehat{P}_{n-1}^\nu(\xi; \varphi) \right) - \nu \widehat{P}_n^\nu(\xi; \varphi). \end{aligned} \quad (100)$$

From (100) and (76) it follows that (for any values of φ) the operator which is unitary equivalent to the "coordinate" operator in the space \mathcal{H}^P acts in not natural way (it is not the operator of multiplication on independent variable). Only for $\xi = \frac{\pi}{2} + k\pi$ we have

$$\widehat{V} \xi \widehat{V}^{-1} = (-i\xi - \nu). \quad (101)$$

Therefore in what follows we shall consider the operator \widehat{V} only with $\xi = \frac{\pi}{2}$. Further we have

$$\begin{aligned} & \widehat{V} e^{\partial_\xi} \widehat{V}^{-1} \widehat{P}_n^\nu(\xi, \varphi) = \widehat{V} e^{\partial_\xi} \widehat{M}_n^-(\xi, 2\nu, e^{-2i\varphi}) \\ & = \widehat{V} \widetilde{M}_n^-(\xi + 1, 2\nu, e^{-2i\varphi}) = (-1)^n \widetilde{M}_n^-(i(\xi + 1) - \nu, 2\nu, e^{-2i\varphi}) \\ & = (-1)^n e^{i\partial_\xi} \widetilde{M}_n^-(i\xi - \nu, 2\nu, e^{-2i\varphi}) = e^{i\partial_\xi} \widehat{P}_n^\nu(\xi, \varphi). \end{aligned}$$

Similarly, we receive

$$\widehat{V} e^{-\partial_\xi} \widehat{V}^{-1} = e^{-i\partial_\xi}.$$

Finally, we have

$$\widehat{V} \xi \widehat{V}^{-1} = -i\xi - \nu, \quad \widehat{V} e^{\pm \partial_\xi} \widehat{V}^{-1} = e^{\pm i\partial_\xi}. \quad (102)$$

From these relations, taking into account (40) - (44), we get an explicit realization of operators

K_+^P, K_-^P, H^P as difference operators in \mathcal{H}^P

$$k_+^P = K_+^P \Big|_{\varphi=\frac{\pi}{2}} = \frac{i}{2}(\nu - i\xi)e^{i\partial_\xi} - \frac{i}{2}(\nu + i\xi)e^{-i\partial_\xi} + \xi, \quad (103)$$

$$k_-^P = K_-^P \Big|_{\varphi=\frac{\pi}{2}} = -\frac{i}{2}(\nu - i\xi)e^{i\partial_\xi} + \frac{i}{2}(\nu + i\xi)e^{-i\partial_\xi} + \xi, \quad (104)$$

$$h^P = H^P \Big|_{\varphi=\frac{\pi}{2}} = \frac{\nu - i\xi}{2}e^{i\partial_\xi} + \frac{\nu + i\xi}{2}e^{-i\partial_\xi}. \quad (105)$$

These operators act on normalized Meixner - Pollaczek polynomials according to

$$K_+^P \hat{P}_n^\nu = -\mu(n) \hat{P}_{n+1}^\nu, \quad K_-^P \hat{P}_n^\nu = -\mu(n-1) \hat{P}_{n-1}^\nu, \quad H^P \hat{P}_n^\nu = (n + \nu) \hat{P}_n^\nu. \quad (106)$$

To compare our results with the ones from the work of N.M.Atakishiev and S.K.Suslov [40], we introduce three auxiliary spaces. First of them - the Hilbert space $\mathring{\mathcal{H}}^P = L^2(d\xi)$ with the basis

$$\left\{ \Phi_n^\nu(\xi, \varphi) = g(\nu; \xi) \hat{P}_n^\nu(\xi, \varphi) \right\}_{n=0}^\infty, \quad (107)$$

where

$$g(\nu; \xi) = \frac{|\Gamma(\nu - i\xi)|2^\nu}{\sqrt{2\pi}\Gamma(n)}. \quad (108)$$

Using the unitary operator

$$W : \mathcal{H}^P \rightarrow \mathring{\mathcal{H}}^P, \quad W P_n^\nu = g \hat{P}_n^\nu = \Phi_n^\nu, \quad (109)$$

and inverse of it

$$W^{-1} : \mathring{\mathcal{H}}^P \rightarrow \mathcal{H}^P, \quad W^{-1} \Phi_n^\nu = g^{-1} \hat{P}_n^\nu = P_n^\nu, \quad (110)$$

we define operators

$$\mathring{K}_+^P = W K_+^P W^{-1}, \quad \mathring{K}_-^P = W K_-^P W^{-1}, \quad \mathring{H}^P = W H^P W^{-1}. \quad (111)$$

Because

$$W e^{i\partial_\xi} W^{-1} = |\nu - 1 + i\xi| e^{i\partial_\xi}, \quad W e^{-i\partial_\xi} W^{-1} = \frac{1}{|\nu + i\xi|} e^{-i\partial_\xi}, \quad (112)$$

and taking into account relations (103)-(105), we obtain the explicit form for operators (111)

at $\varphi = \frac{\pi}{2}$

$$k_+^P = \mathring{K}_+^P \Big|_{\varphi=\frac{\pi}{2}} = \frac{i}{2}(\nu - i\xi)|\nu - 1 + i\xi| e^{i\partial_\xi} - \frac{i}{2} \frac{\nu + i\xi}{|\nu + i\xi|} e^{-i\partial_\xi} + \xi, \quad (113)$$

$$k_-^P = \mathring{K}_-^P \Big|_{\varphi=\frac{\pi}{2}} = -\frac{i}{2}(\nu - i\xi)|\nu - 1 + i\xi| e^{i\partial_\xi} + \frac{i}{2} \frac{\nu + i\xi}{|\nu + i\xi|} e^{-i\partial_\xi} + \xi, \quad (114)$$

$$h^P = \mathring{H}^P \Big|_{\varphi=\frac{\pi}{2}} = \frac{1}{2}(\nu - i\xi)|\nu - 1 + i\xi| e^{i\partial_\xi} + \frac{1}{2} \frac{\nu + i\xi}{|\nu + i\xi|} e^{-i\partial_\xi}. \quad (115)$$

Let us denote by $\overset{\circ}{\mathcal{H}}_A^P$ the space $\overset{\circ}{\mathcal{H}}^P$ with the following choice of parameters

$$\varphi = \frac{\pi}{2}, \quad \xi = \frac{x}{\lambda}, \quad \nu(\nu - 1) = \lambda^{-4}. \quad (116)$$

Let us choose in the space $\overset{\circ}{\mathcal{H}}_A^P$ a new basis

$$\overset{\circ}{\Psi}_n = S\Phi_n^\nu = \Phi_n^\nu e^{i\arg\Gamma(\nu+i\xi)}, \quad n = 0, 1, 2, \dots \quad (117)$$

By the action of the unitary operator $S = e^{i\arg\Gamma(\nu+i\xi)}$ the difference operators $e^{\pm i\partial_\xi}$ are transformed according to

$$\begin{aligned} Se^{i\partial_\xi}S^{-1} &= e^{i\arg\Gamma(\nu+i\xi)}e^{i\partial_\xi}e^{-i\arg\Gamma(\nu+i\xi)} = \\ &= e^{i\arg(\nu-1+i\xi)}e^{i\arg\Gamma(\nu+i\xi)}e^{-i\arg\Gamma(\nu+i\xi)}e^{i\partial_\xi} = e^{i\arg(\nu-1+i\xi)}e^{i\partial_\xi}; \end{aligned} \quad (118)$$

$$Se^{-i\partial_\xi}S^{-1} = e^{i\arg\Gamma(\nu+i\xi)}e^{-i\partial_\xi}e^{-i\arg\Gamma(\nu+i\xi)} = e^{i\arg(\nu-i\xi)}e^{-i\partial_\xi}. \quad (119)$$

Then Hamiltonian $\overset{\circ}{H}^P$ (115) became

$$\overset{\circ}{h}_S^P = S\overset{\circ}{h}^P S^{-1} = \frac{1}{2}(\nu(\nu - 1) + i\xi + \xi^2)e^{i\partial_\xi} + \frac{1}{2}e^{-i\partial_\xi}. \quad (120)$$

In the space $\overset{\circ}{\mathcal{H}}_A^P$ Hamiltonian (120) takes the form

$$\overset{\circ}{h}_A^P = \frac{1}{2}\left(\frac{1}{\lambda^4} + i\frac{x}{\lambda} + \frac{x^2}{\lambda^2}\right)e^{i\lambda\partial_x} + \frac{1}{2}e^{-i\lambda\partial_x}. \quad (121)$$

To check that our Hamiltonian coincides with Hamiltonian from the work [40] it is necessary to pass to the space \mathcal{H}_A^P by unitary transformation $V_A : \overset{\circ}{\mathcal{H}}_A^P \rightarrow \mathcal{H}_A^P$, such that

$$\phi_n^A = V_A \overset{\circ}{\phi}_n = \eta(x) \overset{\circ}{\phi}_n, \quad (122)$$

where

$$\eta(x) = \lambda^{2i\frac{x}{\lambda} - \frac{1}{2}}, \quad \overset{\circ}{\phi}_n = \Phi_n^\nu\left(\frac{x}{\lambda}, \frac{\pi}{2}\right)e^{i\arg\Gamma(\nu+i\frac{x}{\lambda})}. \quad (123)$$

Under this transformation we have

$$V_A e^{i\lambda\partial_x} V_A^{-1} = \eta(x) e^{i\lambda\partial_x} \eta^{-1}(x) = \frac{\eta(x)}{\eta(x+i\lambda)} e^{i\lambda\partial_x} = \lambda^2 e^{i\lambda\partial_x}; \quad (124)$$

$$V_A e^{-i\lambda\partial_x} V_A^{-1} = \eta(x) e^{-i\lambda\partial_x} \eta^{-1}(x) = \frac{\eta(x)}{\eta(x-i\lambda)} e^{-i\lambda\partial_x} = \lambda^{-2} e^{-i\lambda\partial_x}. \quad (125)$$

Finally, we have

$$\begin{aligned} h_A^P &= V_A \overset{\circ}{h}_A^P V_A^{-1} = \frac{1}{2}\left(\frac{1}{\lambda^2} + i\lambda x + x^2\right)e^{i\lambda\partial_x} + \frac{1}{2\lambda^2}e^{-i\lambda\partial_x} = \\ &= \frac{1}{\lambda^2}\text{ch}(i\lambda\partial_x) + \frac{1}{2}(x+i\lambda)x e^{i\lambda\partial_x}, \end{aligned} \quad (126)$$

that coincides with Hamiltonian of linear relativistic oscillator from the work [40] (see the formula (4.1) in this work).

6.3 Coherent states for Meixner - Pollaczek oscillator

We shall restrict ourself to construction of Barut - Girardello coherent states for Meixner - Pollaczek oscillator in the space \mathcal{H}^P at $\varphi = \frac{\pi}{2}$. We have

$$k_-^P |z\rangle = z|z\rangle. \quad (127)$$

The series representation of the coherent state $|z\rangle$ by the Fock basis $\left\{|z\rangle = \widehat{P}_n^\nu(\xi, \frac{\pi}{2})\right\}_{n=0}^\infty$ in the space \mathcal{H}^P looks like

$$|z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{(\mu(n-1))!} |n\rangle, \quad \mu(n) = \sqrt{(n+1)(n+2\nu)} \quad (128)$$

$$\mathcal{N}^2(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{((\mu(n-1))!)^2} = (2\nu)_n \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n! \Gamma(n+2\nu)}. \quad (129)$$

The radius of convergence of the series in (129) is equal to $R = \infty$. Taking into account (54), we obtain

$$\mathcal{N}^2(|z|^2) = \frac{\Gamma(2\nu)}{|z|^{2\nu-1}} I_{2\nu-1}(2|z|). \quad (130)$$

As result we have

$$\begin{aligned} |z\rangle &= \frac{|z|^{\nu-\frac{1}{2}}}{\sqrt{\Gamma(2\nu) I_{2\nu-1}(2|z|)}} \sum_{n=0}^{\infty} (-1)^n \frac{\widehat{P}_n^\nu(\xi, \frac{\pi}{2})}{\sqrt{n! (2\nu)_n}} z^n = \\ &= \frac{|z|^{\nu-\frac{1}{2}}}{\sqrt{\Gamma(2\nu) I_{2\nu-1}(2|z|)}} \sum_{n=0}^{\infty} (-1)^n \frac{\widetilde{M}_n^-(i\xi - \nu, 2\nu; -1)}{\sqrt{n! (2\nu)_n}} z^n. \end{aligned} \quad (131)$$

Then from the relation

$$\widetilde{M}_n^-(i\xi - \nu, 2\nu; -1) = \sqrt{\frac{(2\nu)_n}{n!}} M_n(i\xi - \nu, 2\nu; -1),$$

we obtain

$$\begin{aligned} |z\rangle &= \frac{|z|^{\nu-\frac{1}{2}}}{\sqrt{\Gamma(2\nu) I_{2\nu-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{M_n(i\xi - \nu, 2\nu; -1)}{\sqrt{n!}} (iz)^n = \\ &= \frac{|z|^{\nu-\frac{1}{2}}}{\sqrt{\Gamma(2\nu) I_{2\nu-1}(2|z|)}} e^{-iz} {}_1F_1\left(\begin{matrix} i\xi - \nu \\ 2\nu \end{matrix} \middle| -2iz\right). \end{aligned} \quad (132)$$

An overlapping of two coherent states is defined by a relation

$$\langle z_1 | z_2 \rangle = I_{2\nu-1}(2\sqrt{z_1 z_2}) [I_{2\nu-1}(2|z_1|) I_{2\nu-1}(2|z_2|)]^{-\frac{1}{2}}. \quad (133)$$

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